Evolution of a modulated KP soliton

1991 J. Phys. A: Math. Gen. 24 1161

(http://iopscience.iop.org/0305-4470/24/6/011)

J. Phys. A: Math. Gen. **24** (1991) 1161–1184. Printed in the UK

Evolution of a modulated KP soliton

Russell L Herman

Department of Mathematical Sciences, University of North Carolina at Wilmington, Wilmington, NC 28403-3297, USA

Received 15 May 1990

Abstract. We investigate the propagation of an initial profile consisting of a planar KP soliton with some small modulations. Using the solution of the Cauchy problem for the linearized KP equation, we find that for large times the modulations move away from the peak of the profile, leaving behind a stable soliton. A generalization of this method is formulated for the study of the stability of solutions of other integrable equations.

1. Introduction

Washimi and Taniuti (1966) derived the Korteweg–deVries (KdV) equation as the equation governing the propagation of small-amplitude ion-acoustic waves in one dimension, using the reductive perturbation technique. Nine years later Kako and Rowlands (1976) applied these methods to the study of two-dimensional perturbations. In their paper they derived the Kadomtsev–Petviashvili equation as a two-dimensional generalization of the KdV equation. This equation was first introduced by Kadomtsev and Petviashvili (1970) in the study of the stability of transverse perturbations of the KdV equation.

The KP equation takes the form

$$(u_t + 6uu_x + u_{xxx})_x + \alpha u_{yy} = 0. \tag{1}$$

It possesses the one-soliton solution

$$u(x,y,t) = 2k^2 \operatorname{sech}^2(kx + \ell y - \omega t) \tag{2}$$

with the dispersion relation

$$\omega = 4k^2 + \frac{\alpha \ell^2}{k}.$$

There have been numerous experimental investigations of the evolution of these two-dimensional ion-acoustic solitons (Gabl *et al* 1984, Lonngren 1983, Lonngren *et al* 1983). These experiments are performed by placing various grids, which are made from a wire mesh, into a carefully prepared plasma region. By applying a voltage across the grid, a density perturbation is created, which depends on the geometry of the wire mesh. At distances far from the grid, the plasma fluid equations can be used, as the hypotheses for this model are satisfied.

0305-4470/91/061161+24$03.50

To get a planar KP soliton, a rectangular uniform mesh is used. Chang (1986), Chang *et al* (1986) *wrinkled* the mesh before inserting it into the plasma device. They observed that initially the amplitude of the profile was irregular; however, at distances far from the grid, i.e. for longer times, they found the profile was independent of the launching structure, and looked like an undisturbed planar soliton.

The authors sought to understand this behaviour through both analytical and numerical work. Numerically, they integrated the KP equation, using a planar soliton with a modulated phase as an initial condition. In the numerical result, the modulations were seen to be damped out, which supported the experimental results.

To understand these results analytically, Chang *et al* (1986) turn to an analysis that is similar to that of Kadomtsev and Petviashvili, which we review in appendix 1. They found that for $u(x,y,t) = 2k^2\mathrm{sech}^2 k(x-x_0(y,t))$ the phase x_0 obeys an equation valid to third order,

$$x_{0tt} - \alpha x_{0yy} + \beta x_{0yyt} = 0 \tag{3}$$

where α and β are positive constants. They stated without proof that an approximate solution to this equation is of the form

$$x_0 \simeq \mathrm{e}^{-\gamma t} f(y \pm \alpha^{1/2} t) \tag{4}$$

displaying the decay sought by these researchers.

However, there is a problem in using singular perturbation theory for the current problem, as we explain in appendix 1. In the above analyses it was assumed that the amplitude and width of the soliton did not vary in the transverse direction. To the order being considered, we show that this implies

$$x_{0yyy} = 0 \tag{5}$$

from the results of the perturbation theory. Therefore, x_0 must be of the form

$$x_0 = ay^2 + by + c \tag{6}$$

which is not sufficient for studying variations far from $y = 0$, as one needs for the current problem.

We will reformulate the problem, not as a problem in singular perturbation theory, but as a Cauchy problem for the KP equation. The best approach would be to apply the inverse scattering method (Ablowitz *et al* 1983, Fokas and Ablowitz 1983) to this problem; however, the only type of initial conditions that can be used in this method are those which go to zero as $x^2 + y^2 \to \infty$. We instead convert the problem to a Cauchy problem for the linearized KP equation, following the work of Burtsev (1985). We will then show how the modulations evolve for large times.

We will end this paper by discussing a general approach to studying the stability of solutions of the KP equation. The solutions, which can be handled by Burtsev's method, are essentially KdV solutions and are not as interesting as many of the new types of solutions, which have been found recently for the KP equation, as well as other two-dimensional integrable evolution equations (Boiti *et al* 1988).

2. Cauchy problem for the linearized KP equation

We are faced with the following problem: given an initial condition close to the planar soliton

$$u_0(t=0) = 2\nu^2 \mathrm{sech}^2 \nu x \tag{7}$$

how will it evolve in time, assuming that its evolution is governed by the KP equation in the form

$$(u_t + 6uu_x + u_{xxx})_x + 3\beta^2 u_{yy} = 0? \tag{8}$$

We will assume that the initial condition takes the form

$$u(t=0) = A_0 \mathrm{sech}^2(\phi + \epsilon\phi_1) \qquad A_0 = 2\nu^2 \qquad \phi = \nu x \tag{9}$$

where ϵ is a small parameter. We expand the initial condition in a Taylor series

$$u(t=0) = A_0 v + \epsilon A_0 \phi_1 v_\phi + \mathrm{O}(\epsilon^2) \qquad v \equiv \mathrm{sech}^2\phi \tag{10}$$

and we expand u about the one-soliton solution:

$$u = u_0 + \epsilon u_1 + \mathrm{O}(\epsilon^2) = 2\nu^2 \mathrm{sech}^2 \nu(x - 4\nu^2 t) + \epsilon u_1. \tag{11}$$

Inserting this into the KP equation, we find that we can pose the problem as an initial value problem for the linearized KdV equation:

$$\begin{aligned} &(u_{1t} + 6(u_0 u_1)_x + u_{1xxx})_x + 3\beta^2 u_{1yy} = 0 \\ &u_1(t=0) = A_0 \phi_1 v_\phi. \end{aligned} \tag{12}$$

Thus, we only need to solve a linear Cauchy problem.

In the following we will transform the x variable to $x' = x - 4\nu^2 t$ and drop the prime. This leads to the equation

$$(u_{1t} - 4\nu^2 u_{1x} + 6(u_0 u_1)_x + u_{1xxx})_x + 3\beta^2 u_{1yy} = 0. \tag{13}$$

Burtsev (1985) solved this problem. In his analysis he studied the large-time behaviour of the Green function $G(x', p; x, t)$ for $x' = x = 0$ and $p/\nu^2 \ll 1$, where p is the associated spectral variable involved in a Fourier transform with respect to y. He found that the solution is described by weakly damped soliton oscillations. In the following we follow Burtsev's analysis for setting up the solution to (13) and examine the initial condition with $\phi_1(y) = \mathrm{e}^{\mathrm{i}\omega y}$. However, the current study differs from Burtsev's in that we do not restrict the analysis by throwing away contributions for $x' \neq 0$ and limiting ourselves to the long transverse modes $p/\nu^2 \ll 1$. We obtain the solution to the Cauchy problem for the linearized KP equation and study the large-t behaviour for arbitrary p/ν^2 and look at the evolution of the initial modulations as they propagate away from the soliton peak.

We first note that a solution to the linearized KP equation can be given by Burtsev (1985)

$$u_1(x,y,t) = e^{-i\Omega t + ipy} \frac{d}{dx}\left[e^{i(k+n)x}\left(1 - \frac{2\nu}{(\nu - ik)(1+e^{2\nu x})}\right)\left(1 - \frac{2\nu}{(\nu - in)(1+e^{2\nu x})}\right)\right]$$
$$= e^{-i\Omega t + ipy} \frac{d}{dx}[\psi(k,x)\psi(n,x)] \tag{14}$$

provided

$$-i\Omega = 4i[n^3 + k^3 + \nu^2(n+k)] \tag{15}$$
$$i\beta p = n^2 - k^2. \tag{16}$$

Here $\psi(k,x)$ is a Jost solution for the Schrödinger equation, $\psi'' + (k^2 + u_0)\psi = 0$.

From the relation (16) only two of the parameters (n,k,p) are independent. Burtsev introduces a parameter z, defined by

$$k(z,p) = z - \frac{ip}{4z} \tag{17}$$
$$n(z,p) = z + \frac{ip}{4z} \tag{18}$$

to get rid of the third parameter. Inserting this transformation into (15) and (16) we have

$$n^2 - k^2 = \left(z + \frac{ip}{4z}\right)^2 - \left(z - \frac{ip}{4z}\right)^2 = ip$$

and

$$-i\Omega(z,p) = 8i\left(z^3 + \nu^2 z - \frac{3p^2}{16z}\right). \tag{19}$$

The general solution is a linear combination of these solutions:

$$u_1(x,y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dp\, r(z,p) e^{-i\Omega(z,p)t + ipy} \frac{d}{dx}[\psi(k,x)\psi(n,x)] \tag{20}$$

where the integral over z must be done with care about the essential singularity $z = 0$. Here $r(z,p)$ is a generalized Fourier coefficient, which is to be determined.

The initial condition can now be used to obtain the function $r(z,p)$. Defining the Fourier transform of the initial condition by

$$u_1(x,p,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipy} u(x,y,0)\, dy \tag{21}$$

then evaluating (20) at $t = 0$, and Fourier transforming with respect to y, we have

$$u_1(x,p,0) = \int_{-\infty}^{\infty} r(z,p) \frac{d}{dx}[\psi(k,x)\psi(n,x)]\, dz. \tag{22}$$

Now we need to get $r(z,p)$ out from under the integral sign. This can be done by using an orthogonality relation. Defining

$$\Psi(z,p,x) \equiv \psi(k,x)\psi(n,x) \tag{23}$$

and

$$\Phi(z,p,x) \equiv \frac{1}{2\pi \mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}[\psi(k,x)\psi(n,x)] \tag{24}$$

from the functions defined in (14), Burtsev proves the orthogonality relation

$$\langle \Phi(z)|\Psi(-z')\rangle \equiv \int_{-\infty}^{\infty} \Phi(z,p,x)\Psi(-z',p,x)\,\mathrm{d}x = z'\delta(z-z'). \tag{25}$$

Thus, multiplying both sides of (22) by $\Psi(-z)$ and integrating with respect to x, one obtains

$$r(z,p) = \frac{1}{2\pi \mathrm{i} z}\int_{-\infty}^{\infty} u_1(x,p,0)\Psi(-z,p,x)\,\mathrm{d}x \equiv \frac{1}{2\pi \mathrm{i}}\langle u_1(x,p,0)|\Psi(-z,p,x)\rangle. \tag{26}$$

Thus, the general solution to the Cauchy problem for the linearized KP equation (13) is found to be

$$u_1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{z}\int_{-\infty}^{\infty}\mathrm{d}p\langle u_1(x',p,0)|\Psi(-z,p,x')\rangle\Phi(z,p,x)\mathrm{e}^{-\mathrm{i}\Omega t+\mathrm{i}py}. \tag{27}$$

3. Behaviour of the peak, $x = 0$

At this point we will deviate from the analysis which Burtsev provides. However, there will later be some similarities in the types of integrals, which we will have to compute. Burtsev rewrites the solution (27) in a form which defines the appropriate Green function:

$$u_1(x,p,t) = \int_{-\infty}^{\infty} u_1(x',p,0)G(x',p;x,t)\,\mathrm{d}x' \tag{28}$$

where the Green function is given by

$$G(x',p;x,t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}z}{z}\mathrm{e}^{-\mathrm{i}\Omega t}\Psi(-z,p,x')\Phi(z,p,x). \tag{29}$$

For the rest of the paper Burtsev computes the asymptotic behaviour of this Green function as $t \to \infty$, for the special case of $x = 0$, the soliton peak, and for $x' = 0$. In his analysis, he restricts his attention to the approximation $p/\nu^2 \ll 1$, but states that the results can be easily generalized to arbitrary values of these parameters. In the following we will not make all of these restrictions. For simplicity we first will focus our attention on the behaviour of the soliton peak, $x = 0$, and reserve the study of general x for the next section.

We consider a specific initial condition, which corresponds to a modulation of the phase in (10). Namely, we will let $\phi_1(y) = \mathrm{e}^{\mathrm{i}\omega y}$. Then from equations (9) and (10) we will solve the linearized KP equation (13) subject to the condition

$$u_1(x,y,0) = A_0 \mathrm{e}^{\mathrm{i}\omega y} (\operatorname{sech}^2\phi)_\phi \qquad A_0 = 2\nu^2 \qquad \phi = \nu x. \tag{30}$$

From equation (21) we compute the Fourier transform of this initial condition as

$$u_1(x,p,0) = \sqrt{2\pi} A_0 \delta(p-\omega)(\operatorname{sech}^2\phi)_\phi. \tag{31}$$

Noting that for $z \to -z$, we have $(k,n) \to (-k,-n)$, then we can compute $\langle u(x',p,0)|\Psi(-z,\omega,x')\rangle$ to obtain

$$\langle u_1(x',p,0)|\Psi(-z,\omega,x')\rangle = -\frac{\sqrt{2\pi}A_0 \mathrm{i}\pi p^2 \delta(p-\omega)}{4\nu^3(\nu+\mathrm{i}k)(\nu+\mathrm{i}n)\sinh(\pi z/\nu)}. \tag{32}$$

Writing $\Phi(z,\omega,x)$ as

$$\Phi(z,\omega,x) = 2\mathrm{i}z\psi(k,x)\psi(n,x) + \nu^2\operatorname{sech}^2\phi\left(\frac{\mathrm{e}^{\mathrm{i}kx}}{\nu-\mathrm{i}k}\psi(n,x) + \frac{\mathrm{e}^{\mathrm{i}nx}}{\nu-\mathrm{i}n}\psi(k,x)\right) \tag{33}$$

and evaluating u_1 at the peak, $x=0$, we have

$$\Phi(z,\omega,0) = -\frac{2\mathrm{i}z(kn+\nu^2)}{(\nu-\mathrm{i}k)(\nu-\mathrm{i}n)}. \tag{34}$$

Putting all of this into equation (27), then for $x=0$ we have the solution for this particular initial value problem along the peak of the soliton

$$u_1(0,y,t) = -\frac{A_0\omega^2\pi\mathrm{e}^{\mathrm{i}\omega y}}{2\nu^3}\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\Omega(z,\omega)t} \frac{kn+\nu^2}{(k^2+\nu^2)(n^2+\nu^2)\sinh(\pi z/\nu)}. \tag{35}$$

We now want to study the integral in (35) for large times. We define this integral as

$$I \equiv \int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{-\mathrm{i}\Omega(z,\omega)t} \frac{kn+\nu^2}{(k^2+\nu^2)(n^2+\nu^2)\sinh(\pi z/\nu)}. \tag{36}$$

The approach that we now take will be to study the pole structure of the integrand and use a stationary phase analysis to determine the appropriate contours to use in the complex z-plane.

First, we note that for $t>0$, we want to close the contour in the UHP (upper-half z-plane). We must first require $\Re(-\mathrm{i}\Omega) < 0$. Letting $z = x + \mathrm{i}y$ in (19), we find that

$$-\mathrm{i}\Omega = 8y\left(-3x^2+y^2-\nu^2-\frac{3\omega^2}{16}\frac{1}{x^2+y^2}\right) + 8\mathrm{i}x\left(-3y^2+x^2+\nu^2-\frac{3\omega^2}{16}\frac{1}{x^2+y^2}\right) \tag{37}$$

thus, we must require

$$\Re(-\mathrm{i}\Omega) = 8y\left(-3x^2+y^2-\nu^2-\frac{3\omega^2}{16}\frac{1}{x^2+y^2}\right) < 0. \tag{38}$$

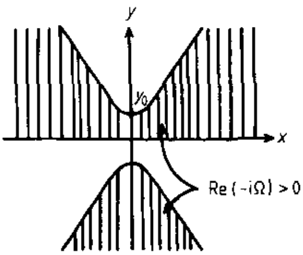

Figure 1. Complex z-plane.

In figure 1 we have sketched the regions defined by (38). We need to find a contour in the upper-half plane, which lies between the x-axis and the upper curve, $\Re(-\mathrm{i}\Omega) = 0$. The point at which this curve crosses the y-axis, $(0, y_0)$, is given by

$$y_0^2 = \tfrac{1}{2}[\nu^2 + (\nu^4 + \tfrac{3}{4}\omega^2)^{1/2}]. \tag{39}$$

Now, we turn to the poles. From the sinh factor in the integrand in the integral in (36), we have an infinite number of poles lying on the imaginary axis:

$$z = \pm \mathrm{i}m\nu \qquad m = 0, 1, 2, \ldots. \tag{40}$$

Of these, only a finite number will contribute to the integral, as only a few of these will lie in the allowed region (38) for $y > 0$. From the definitions of k and n, we find that the only pole at $z = 0$ is from the sinh factor. However, we will later show that this is really not a pole, due to the $1/z$ factor in the definition of Ω.

The other poles come from $k = \pm\mathrm{i}\nu$, and $n = \pm\mathrm{i}\nu$. Using the relations (17) and (18), we have

$$k = \pm\mathrm{i}\nu \quad \Rightarrow \quad z - \mathrm{i}\omega/4z = \pm\mathrm{i}\nu$$

$$n = \pm\mathrm{i}\nu \quad \Rightarrow \quad z + \mathrm{i}\omega/4z = \pm\mathrm{i}\nu$$

which can be solved to give

$$k = \pm\mathrm{i}\nu: \qquad z = \pm\tfrac{1}{2}[\mathrm{i}\nu \mp (\mathrm{i}\omega - \nu^2)^{1/2}] \tag{41}$$

$$n = \pm\mathrm{i}\nu: \qquad z = \pm\tfrac{1}{2}[\mathrm{i}\nu \mp (-\mathrm{i}\omega - \nu^2)^{1/2}]. \tag{42}$$

Thus, we have a total of eight poles from these contributions, half of which will lie in the upper-half plane.

We now need to choose a convenient closed contour for which we can study the large-t behaviour. Following Burtsev, we now look at the stationary phase points. Once we find these, we can then require the contour to pass through them, and will be able to evaluate the behaviour along the contour using the method of steepest descents.

We obtain the saddle points z_j from $f'(z_j) = 0$, where

$$f(z) = -\mathrm{i}\Omega = 8\mathrm{i}\left(z^3 + \nu^2 z - \frac{3\omega^2}{16z}\right). \tag{43}$$

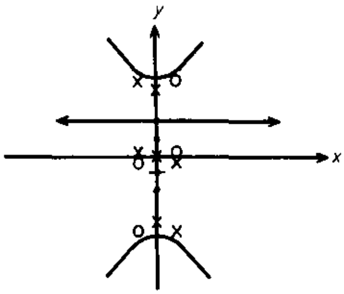

Figure 2. Location of poles and stationary phase points for $\gamma < 2/3$.

Thus, we need to solve the equation

$$3z_j^4 + \nu^2 z_j^2 + C = 0 \qquad C \equiv \frac{3\omega^2}{16} \tag{44}$$

obtaining the four saddle points

$$z_j = (-1)^{j+1}\frac{\mathrm{i}}{\sqrt{6}}[\nu^2 + (-1)^m(\nu^4 - 12C)^{1/2}]^{1/2} \qquad j = 1,2,3,4 = \begin{cases} 2m \\ 2m+1. \end{cases} \tag{45}$$

Here we have defined m by $m = \frac{1}{2}(j - j \bmod 2)$.

Now that we know where the saddle points are, we have to find the paths of steepest descent. This is done in detail in appendix 2. The results depend on the parameter $\gamma = \omega/\nu^2$.

If $\gamma < 2/3$, then we find that the extremal paths are given by

$$\alpha = (2p+1)\frac{\pi}{2} - \frac{\theta}{2} = \begin{cases} p\pi & \text{m+j even} \\ p\pi + \frac{\pi}{2} & \text{m+j odd} \end{cases} \qquad (p = 0, 1) \tag{A2.15}$$

where θ is defined as

$$\theta = \begin{cases} \pi & \text{m+j even} \\ 0 & \text{m+j odd.} \end{cases} \tag{A2.14}$$

Using this information, the following table is constructed where we denote '-1' as a steepest descent path, and '$+1$' as the steepest ascent path, which we must avoid.

Table 1. Paths for $\gamma < \frac{2}{3}$.

j	m	$\alpha = 0$	$\alpha = \frac{\pi}{2}$	Plane for z_j
1	0	+1	−1	UHP
2	1	+1	−1	LHP
3	1	−1	+1	UHP
4	2	−1	+1	LHP

In figure 2 we show a sketch of the complex z-plane with the critical points for $\gamma = 0.5$. The poles for $k = \pm\mathrm{i}$ are denoted by crosses (not on the axis), and the poles corresponding to $n = \pm\mathrm{i}$ are denoted by open circles. The poles from the sinh are given by small crosses along the imaginary axis, of which only three appear in this figure. Finally, the stationary points are dentoted by small full circles. In this figure two of these have horizontal lines through them, indicating the direction of steepest descent, where the long line denotes the steepest descent curve for this problem. These stationary phase points are obtained from z_1 and z_3 in the above table.

With this information we can choose the line parallel to the real axis, going through the upper stationary point in the UHP. To close the contour we connect this line at infinity to the real axis. Using this contour to evaluate the integral, we find that only two poles are enclosed by the contour. We can add up these contributions using the calculus of residues, and evaluate the asymptotic behaviour of the integral along the steepest descent line using the method of stationary phase.

The paths of steepest descent for the case $\gamma > 2/3$ are given by

$$\alpha = \begin{cases} \frac{1}{2}\phi_j + p\pi & P_j = 0 \\ \frac{1}{2}\phi_j + (2p+1)\frac{\pi}{2} & P_j = 1 \end{cases} \tag{A2.22}$$

where ϕ_j is the phase of the stationary point z_j, $p = 0, 1$, and

$$P_j = \begin{cases} 0 & z_j \text{ in quadrants I, III} \\ 1 & z_j \text{ in quadrants II, IV.} \end{cases} \tag{A2.19}$$

We show in figure 3, for $\gamma => 2/3$ a sketch of the complex z-plane with the poles and stationary points marked. The poles are indicated as before. The major feature, which is different, is the location of the stationary phase points and the direction of steepest descent. These four points are indicated by a small circle, with the steepest descent direction indicated on the points in the upper-half plane. The curve, which has to be added to close the contour, now goes through two stationary phase points. As γ increases, there are more poles included inside the closed contour. In order to investigate the general behaviour of the integral (36), we will compute the effect of each pole as if it were inside the contour. Since we are only interested in the general features of this problem, we will not worry about how to add up all of these contributions, since the sum of these contributions for arbitrary ω/ν^2 is quite different.

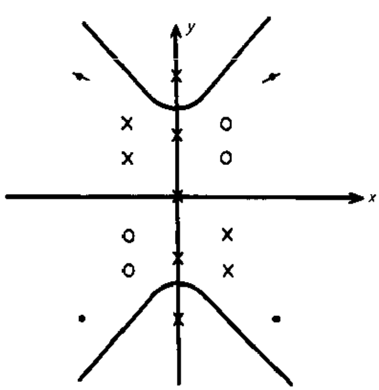

Figure 3. Location of poles and stationary phase points for $\gamma > 2/3$.

We are now ready to consider the integral given in equation (36)

$$I = \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\Omega(z,\omega)t} \frac{kn + \nu^2}{(k^2+\nu^2)(n^2+\nu^2)\sinh(\pi z/\nu)}. \tag{46}$$

In order to obtain the pole contributions from $k = \pm i\nu$, we look at equations (15) and (16)

$$-i\Omega = 4i[n^3 + k^3 + \nu^2(k+n)] \qquad i\omega = n^2 - k^2$$

and equations (17) and (18):

$$k = z - \frac{ip}{4z} \qquad n = z + \frac{ip}{4z}.$$

Inserting $k = \pm i\nu$ in the first set of equations, we find

$$i\omega = n^2 + \nu^2 \qquad -i\Omega = -4\omega n. \tag{47}$$

Adding the second set of equations for k and n, and then solving for n, gives

$$n = 2z_k \mp i\nu. \tag{48}$$

From this information we easily find the residues due to these poles, if they are enclosed by the contour:

$$2\pi i\mathrm{Res}[z = z_k] = \frac{2\pi z_k}{\omega \sinh(\pi z_k/\nu)}. \tag{49}$$

In a similar fashion, we can compute the residue for the zeros of $n = \pm i\nu$:

$$2\pi i\mathrm{Res}[z = z_n] = -\frac{2\pi z_n}{\omega \sinh(\pi z_n/\nu)}. \tag{50}$$

From the argument of the exponential, we have

$$-i\Omega(z_k) = -4\omega n(z_k) \qquad -i\Omega(z_n) = 4\omega k(z_n). \tag{51}$$

Since we have already guaranteed that $\Re(-i\Omega) < 0$, we have that any of these pole contributions would lead to damped oscillations, if they are inside the contour.

The other poles, which could possibly arise, are from the $\sinh(\pi z/\nu)$ factor. If these poles are enclosed in the contour, there would be a finite number. For $m = 0$, we have a slightly more complicated contribution from each m. We evaluate the factors in the integrand:

$$\begin{aligned} kn + \nu^2 &= (1-m)\nu^2 - \frac{\omega^2}{16m^2\nu^2} \\ n^2 + \nu^2 &= (1-m^2)\nu + i\frac{\omega}{4} + \frac{\omega^2}{16\nu^2 m^2} \\ k^2 + \nu^2 &= (1-m^2)\nu - i\frac{\omega}{4} + \frac{\omega^2}{16\nu^2 m^2} \\ -i\Omega &= i(\nu^3 m^3 - \nu^3 m^2 - \frac{3\omega^2}{16m\nu}). \end{aligned} \tag{52}$$

Using the L'Hopital rule we find

$$\lim_{z\to im\nu}\left(\frac{z-im\nu}{\sinh(\pi z/\nu)}\right)=\lim_{z\to im\nu}\left(\frac{\nu}{\pi\cosh(\pi z/\nu)}\right)=\frac{\nu}{\pi\cos(m\pi)}=(-1)^m\frac{\nu}{\pi}. \tag{53}$$

The important factor, of course, is the behaviour of $-\mathrm{i}\Omega t$. As can be seen, this is real and negative, if in the allowed region, $\Re(-\mathrm{i}\Omega)<0$. Thus, these poles lead to a contribution which is damped, and does not have any oscillatory piece.

The last point to check is $z=0$. This point is a zero of the sinh factor; however, we see that as $z\to 0$

$$\frac{kn+\nu^2}{(k^2+\nu^2)(n^2+\nu^2)\sinh(\pi z/\nu)}\longrightarrow \frac{16z\nu(16z^2\nu^2+16z^4+\omega^4)}{\pi(16z^2\nu^2+16z^4-8\mathrm{i}\omega z^2-\omega^2)(16z^2\nu^2+16z^4+8\mathrm{i}\omega z^2-\omega^2)}\longrightarrow 0. \tag{54}$$

So, $z=0$ is not really a pole. We still must be careful near this point, because of the $1/z$ term in $-\mathrm{i}\Omega t$. If we go around $z=0$ in a small semicircle of radius ρ above the origin, and let the radius go to zero, we find

$$I_0\equiv\lim_{\rho\to 0}\int_{-\pi}^{0}\mathrm{d}\theta\, g(\rho,\theta)\exp[8\mathrm{i}t(\rho^3\mathrm{e}^{3\mathrm{i}\theta}+\nu^2\rho\mathrm{e}^{\mathrm{i}\theta}-\tfrac{3}{16}\omega^2\mathrm{e}^{-\mathrm{i}\theta})].$$

However, $g(\rho,\theta)$ vanishes in this limit, and the exponential can be written as

$$\exp[-(3\omega^2t/\rho)(\sin\theta+\mathrm{i}\cos\theta)].$$

Since $t>0$ and $\sin\theta>0$, then as $\rho\to 0$, the integrand tends to zero. Therefore, there is no contribution from this point.

From the Cauchy integral theorem, the integral we desire can be written as

$$\int_{-\infty}^{\infty}=\oint-\int_{-\infty}+\int_{\text{steep desc}}. \tag{55}$$

We still have to get the contribution from the contours. As $|z|\to\infty$, for $\Re(-\mathrm{i}\Omega)<0$, the two extreme contours at infinity vanish, as the integrand goes to zero there. The only contour integral left to evaluate is the steepest descent contour. From the standard results of the method of steepest descents, for large t this integral is given by (Bleistein and Handelsman 1986, Whitham 1974)

$$I\simeq g(z_0)\sqrt{\frac{\pi}{2|f''(z_0)|t}}\exp[-\mathrm{i}\Omega(z_0)t+\mathrm{i}\alpha]\qquad f(z)\equiv-\mathrm{i}\Omega(z) \tag{56}$$

where z_0 denotes the stationary phase point. In our analysis, we have obtained these points and the direction α. In fact, there are two such points on our contour, when $\gamma>\frac{2}{3}$, so we must add both of these contributions. We have from (43)

$$f(z)=8\mathrm{i}(z^3+\nu^2z-3\omega^2/16z) \tag{57}$$

and

$$f''(z) = 8\mathrm{i}(6z - 2C/z^3). \tag{58}$$

So, the contribution from this contour consists of one, or two, terms of the form

$$I \sim \left(\frac{kn_\nu^2}{(k^2+\nu^2)(n^2+\nu^2)\sinh(\pi z/\nu)}\right)(z_0)\sqrt{\frac{\pi}{2|f''(z_0)|t}}\exp[-\mathrm{i}\Omega(z_0)t + \mathrm{i}\alpha]. \tag{59}$$

This completes the analysis of the initial value problem to first order in ϵ. We have found that all the contributions to the first-order solutions are damped. The important contributions are due to the poles, as the stationary phase points behave as

$$t^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\Omega_1 t} \qquad \Omega_1 \equiv \Omega(\text{st phase pt})$$

while the poles due to the $k = \pm\mathrm{i}\nu, n = \pm\mathrm{i}\nu$ lead to damped oscillations of the form

$$\mathrm{e}^{-\mathrm{i}\Omega_2 t} \qquad \Omega_2 \equiv \Omega(\text{pole}).$$

In fact, since we have a relation between the zeros of these two equations, $z_n = -z_k^*$, the damped oscillations can be shown to be of the form

$$A(z_k)\mathrm{e}^{-\mathrm{i}\Omega(z_k)t} + A^*(z_k)\mathrm{e}^{-\mathrm{i}\Omega^*(z_k)t} \tag{60}$$

when these zeros have a positive imaginary part less than one. In this case the decaying oscillations are travelling in opposite directions, but damping at the same rate. Burtsev (1985) has also obtained the same general features for the asymptotic behaviour of the restricted, $\gamma \ll 1$, Green function, as discussed at the beginning of this section.

4. Behaviour for general x

Having obtained the asymptotic behaviour of the peak of the soliton for large times, we can now look at more general values of x. The general methods are basically the same as above. We start with the solution which was obtained in equation (27) as

$$u_1(x,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{z}\int_{-\infty}^{\infty}\mathrm{d}p\langle u_1(x',p,0)|\Phi(-z,p,x')\rangle\Phi(z,p,x)\mathrm{e}^{-\mathrm{i}\Omega t+\mathrm{i}py}. \tag{27}$$

Inserting the result for the inner product $\langle u(x',p,0)|\Psi(-z,p,x')\rangle$, we can rewrite this as

$$\begin{aligned} u_1(x,y,t) &= -\frac{A_0\omega^2}{8\nu^2}\mathrm{e}^{\mathrm{i}\omega y}\partial_\phi\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{z}\mathrm{e}^{\mathrm{i}f(z)}\frac{\nu^2\tanh^2\phi - 2\mathrm{i}z\nu\tanh\phi - kn}{(k^2+\nu^2)(n^2+\nu^2)\sinh(\pi z/\nu)} \\ &\equiv -\frac{A_0\omega^2}{8\nu^2}\mathrm{e}^{\mathrm{i}\omega y}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{z}\mathrm{e}^{\mathrm{i}f(z)}H(z) \end{aligned} \tag{61}$$

where we have defined $f(z)$ by

$$\mathrm{i}f(z) \equiv 8\left(z^3 + \frac{x}{4t}z - \frac{3\omega^2}{16z}\right)t \tag{62}$$

and $H(z)$ by

$$H(z) = \frac{-2\nu^2\tanh^3\phi + 4\mathrm{i}\nu z\tanh^2\phi + 2(\nu^2 + 2z^2)\tanh\phi - 2\mathrm{i}z(kn + \nu^2)/\nu}{(k^2 + \nu^2)(n^2 + \nu^2)\sinh(\pi z/\nu)}. \tag{63}$$

We now study the behaviour of the above integral for large t, keeping x/t fixed. This will give results describing the long time behaviour of waves travelling at the same velocity x/t. In particular, we define the velocity $V = x/4t$ and we look for the stationary phase points again. Comparing $\mathrm{i}f(z)$ in equation (62) with $-\mathrm{i}\Omega(z)$ in equation (19)

$$-\mathrm{i}\Omega = 8\left(z^3 + \nu^2 z - \frac{3\omega^2}{16z}\right)t \tag{19}$$

we see that the analysis of the stationary phase points in the present case is the same as we had already encountered. By replacing ν^2 by V in the previous results, we have the same four stationary phase points:

$$z_j = (-1)^j i\sqrt{\frac{V}{6}}\sqrt{1 + (-1)^{m+1}(1 - \tfrac{9}{4}\gamma^2)^{1/2}} \qquad \gamma \equiv \frac{\omega}{V}. \tag{64}$$

Only the scaling of z_j is changed, and the stationary phase directions can be shown to be unaffected.

In the previous analysis we also had to require that $\Re(-\mathrm{i}\Omega) < 0$. Here we require $\Re(\mathrm{i}f(z)) < 0$. For this we have

$$\Re(-\mathrm{i}f) = 8y\left(-3x^2 + y^2 - V - \frac{3\omega^2}{16}\frac{1}{x^2 + y^2}\right) < 0 \qquad z = x + \mathrm{i}y. \tag{65}$$

This results in the same sketch as before, where

$$y_0^2 = [\tfrac{1}{2}(V + (V^2 + \tfrac{3}{4}\omega^2)^{1/2}]. \tag{66}$$

We see that as $V = \nu^2$ is changed, the boundary $\Re(\mathrm{i}f) = 0$ will move, as well as the stationary phase points.

We can now deal with the poles. The poles in (61) are exactly the same as before. Therefore, there will be different pole contributions for waves moving at significantly different speeds, since there may be more or less poles enclosed by the contour, whose location depends on V. These poles are given by

$$\begin{aligned} &z = \pm\tfrac{1}{2}(\mathrm{i}\nu \mp \sqrt{\mathrm{i}\omega - \nu^2}) \qquad \text{for } k = \pm\mathrm{i}\nu \\ &z = \pm\tfrac{1}{2}(\mathrm{i}\nu \mp \sqrt{-\mathrm{i}\omega - \nu^2}) \qquad \text{for } n = \pm\mathrm{i}\nu \\ &z = \pm\mathrm{i}m\nu \qquad \text{for } \sinh(\pi z/\nu) = 0. \end{aligned} \tag{67}$$

The only possible difference is that $z = 0$ may become a new pole. In the denominator of $H(z)$ in equation (63) we have as z tends to 0 that

$$(k^2 + \nu^2)(n^2 + \nu^2)\sinh(\pi z/\nu) \longrightarrow \frac{\pi\omega^4}{256\nu z^2}. \tag{68}$$

In the numerator we have, as $z \to 0$,

$$-2\nu^2 \tanh^3\phi + 4\mathrm{i}\nu z \tanh^2\phi + 2(\nu^2 + 2z^2)\tanh\phi - 2\mathrm{i}z(kn + \nu^2)/\nu \to -\frac{\mathrm{i}\omega^2}{8\nu z}. \tag{69}$$

Therefore, $H(z)$ in (61) behaves as

$$H(z) \to -\frac{32\mathrm{i}}{\pi\omega^2} z \to 0 \qquad \text{as} \quad z \to 0. \tag{70}$$

So, z is not a pole in this more general case, either.

Putting this information together as before, we find that the asymptotic behaviour of the solution in (61) takes the form

$$u(x, y, t) \sim 2\pi\mathrm{i} \sum_{z_p} \mathrm{Res}\,[\mathrm{e}^{\mathrm{i}f(z)} H(z)]_{z_p} + \sum_{z_s} H(z_s) \sqrt{\frac{\pi}{2f''(z_s)}} \exp[\mathrm{i}f(z_s) + \mathrm{i}\alpha] \tag{71}$$

where

$$f(z) = 8\left(z^3 - \frac{3\omega^2}{16z}\right)t + 2xz \tag{72}$$

$$f''(z) = \frac{3t}{z^3}(16z^4 - \omega^2) \tag{73}$$

the first sum being over the poles, and the second sum being over the stationary phase points.

Since $\Re(\mathrm{i}f) < 0$, we see that each term will decay exponentially for fixed V. As $f''(z)$ is linear in t, the contributions due to the stationary phase points will not last as long as those from the poles. The longest remaining terms will be due to those poles closest to the real axis in the z-plane. These will either be the poles from $k = \pm\mathrm{i}\nu, n = \pm\mathrm{i}\nu$, or from $z = im\nu$. As for the result for the soliton peak, where $V = \nu^2$, we use the fact that the $k = \pm\mathrm{i}\nu$ and $n = \pm\mathrm{i}\nu$ poles are related by $z_n = -z_k^*$. Inserting into u_1 leads to the propagation of oscillations in opposite directions. However, for ω/ν^2 greater than some critical value, the contribution from the pole $z = \mathrm{i}\nu$ will begin to dominate the solution.

For large enough times we can ignore the contribution from the stationary phase points. Investigating the solution for the x-dependence, we see that u_1 can be written in the form

$$\begin{aligned} u_1 &= -\frac{A_0\omega^2}{8\nu^2}\mathrm{e}^{\mathrm{i}\omega y} \int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{\mathrm{i}f(z)} H(z) \\ &\approx -\frac{\mathrm{i}\pi\omega^2}{2}\mathrm{e}^{\mathrm{i}\omega y} \sum_{\text{poles}\, z_p} \lim_{z \to z_p} (z - z_p) H(z) \mathrm{e}^{8\mathrm{i}(z^3 - 3\omega^2/16)t + 2\mathrm{i}zx}. \end{aligned} \tag{74}$$

So, we would expect to see a complicated superposition of oscillations along the x direction. The form of equation (74) can be plotted, in order to examine the effects of this contribution at various times. In figure 4 we plot the initial condition for the values

$$A_0 = 2 \qquad \omega = 5 \qquad \nu = 1. \tag{75}$$

In figures 5–8 we plot parts of the approximate solution in (74), so that we can see how some of the fluctuations in figure 4 propagate away from the centre of the soliton. In the later figures, we see how these oscillations move out and a secondary *wavefront* follows. In time a stable planar soliton is left behind, much in the same way as the numerical simulations of Chang (1986) and Chang *et al* (1986) had appeared.

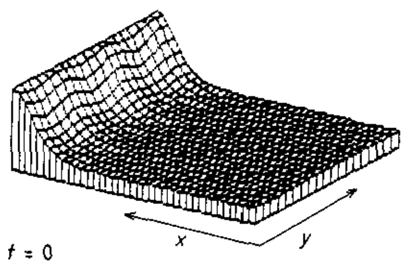

Figure 4. Initial modulated planar soliton ($t = 0$).

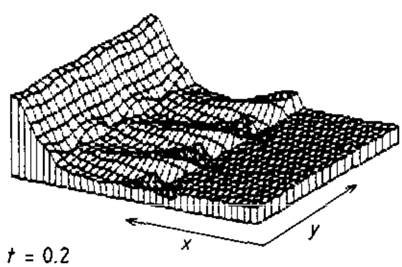

Figure 5. Evolution of modulations due to the pole $z = \mathrm{i}$ according to equation (74) for $A_0 = 2, \omega = 5, \nu = 1$ at $t = 0.2$.

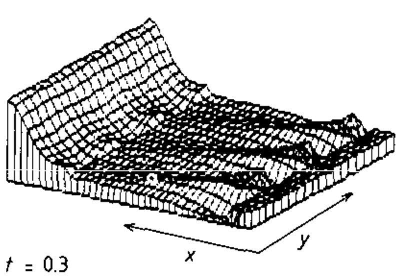

Figure 6. Evolution of modulations at $t = 0.3$.

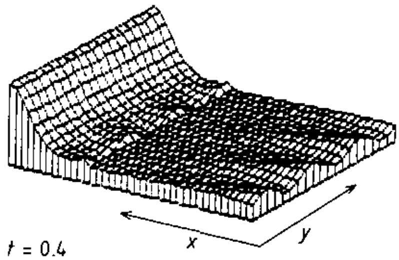

Figure 7. Evolution of modulations at $t = 0.4$.

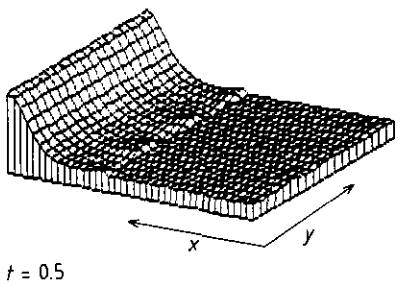

Figure 8. Evolution of modulations at $t = 0.5$.

5. Stability of other KP solutions

In the previous sections we had seen how the solution of the linearized KP equation (12) could be found by writing the solution as a linear combination of the basis states in (14). The unknown expansion coefficients, $r(z,p)$, could then be determined through the use of the initial condition and some orthogonality relations with the adjoint set.

The type of solution considered was of a special form. Namely, it was independent of y. Such solutions are essentially one-dimensional solutions. In recent investigations localized solutions of two-dimensional evolution equations have been found, which deacy exponentially in all directions (Boiti *et al* 1988, Fokas and Santini 1989, Hietarinta 1990). In this section we would like to propose a method for analysing the stability of such solutions. We will do this by drawing a parallel to the method used in section 2.

We first consider finding the eigenstates of the linearized KP operator in equation (12). It is well known that the KdV equation (Ablowitz and Segur 1981)

$$u_t + 6uu_x + u_{xxx} = 0 \tag{76}$$

is a consistency condition for the Lax pair

$$\psi_{xx} + (\lambda^2 + u)\psi = 0 \tag{77}$$

$$\psi_t = -\psi_{xxx} - 3(u - \lambda^2)\psi. \tag{78}$$

Letting $\lambda = n, k$, these equations imply

$$\begin{aligned}(\partial_t + 6u\partial_x + \partial_x^3)\psi(n,x,t)\psi(k,x,t) \\ = 3(n^2 - k^2)(\psi_x(n,x,t)\psi(k,x,t) - \psi(n,x,t)\psi_x(k,x,t)).\end{aligned} \tag{79}$$

Differentiating this with respect to x, and using equation (77), yields

$$(\partial_t + 6\partial_x u + \partial_x^3)(\psi(n,x,t)\psi(k,x,t))_x = -3(n^2-k^2)^2\psi(n,x,t)\psi(k,x,t). \tag{80}$$

Note that these functions differ slightly from those defined in equation (14), as the time dependence has been incorporated into these functions.

Now, the operator for the linearized KP equation is of the form

$$\mathcal{L} = \partial_x(\partial_t + 6\partial_x u_0 + \partial_x^3) + 3\beta^2\partial_y^2 \tag{81}$$

where u_0 is a solution of the KdV equation (76). Defining the functions

$$\Omega(x,y,t;n,k) = \mathrm{e}^{\mathrm{i}py}\partial_x(\psi(n,x,t)\psi(k,x,t)) \tag{82}$$

we find that

$$\mathcal{L}\Omega(x,y,t;n,k) = -3[\beta^2p^2 + (n^2-k^2)]\Omega(x,y,t;n,k). \tag{83}$$

Therefore, the functions $\Omega(x,y,t;n,k)$ are eigenfunctions for the operator $\mathcal{L}$.

In the previous analysis these eigenfunctions were used to determine the solution of the Cauchy problem for the linearized KP equation. The solution was expanded in this basis, using unknown expansion coefficients. These coefficients were then determined by using certain orthogonality relations with the adjoint eigenfunctions. We now turn to the linearized KP equation, where u_0 can depend on y.

In the general case we need the Lax pair, which is associated with the KP equation (Ablowitz *et al* 1983, Fokas and Ablowitz 1983). The KP equation is a consistency condition for the pair of equations

$$\beta\psi_y + \psi_x x + u_0(x,y,t)\psi = 0 \tag{84}$$

$$\psi_t + 4\psi_{xxx} + 6u_0\psi_x + 3\left(u_{0x} - \beta\int_{-\infty}^{x} u_{0y}\mathrm{d}y'\right)\psi + \alpha\psi = 0. \tag{85}$$

Here α is an arbitrary constant. We define $\phi(x,y,t)$ as the solution of the adjoint problem, which can obtained from equations (84) and (85) by letting $\beta \to -\beta$. Using these equations, we find that

$$\mathcal{L}(\psi\phi)_x = -2\alpha(\psi\phi)_x. \tag{86}$$

Again, we have found a set of eigenfunctions of the linearized KP operator. By choosing the solutions $\psi(x,y,t)$ and $\phi(x,y,t)$ to satisfy certain boundary conditions as $|x| \to \infty$, then the value of α can be fixed, depending only on λ. If we can show that these eigenfunctions form a complete set and if we can obtain orthogonality relations between these functions and the corresponding adjoint set, then we can express the solution of the linearized KP equation as a linear combination of these states and solve for the expansion coefficients.

In general, we can obtain the solutions of the above Lax pair, by employing the inverse scattering formalism for this system. However, as has been noted in the literature, this involves the use of the so-called $\bar{\partial}$ problem (Ablowitz *et al* 1983, Fokas and Ablowitz 1983, 1984). Work is currently under way to use this method for the KP and the Davey–Stewartson equations.

Finally, this method has been used to study perturbations of equations in one dimension, through the use of a direct approach (Herman 1988, 1990a, b). In this type of an application we are interested in solving the inhomogeneous problem

$$\mathcal{L}u = \mathcal{F}. \tag{87}$$

Here $\mathcal{L}$ is the linearized evolution operator and $\mathcal{F}$ is a driving term, resulting from the perturbation expansion. In the general method the eigenfunctions of the linearized operator are found by through a transformation of the original Lax pair to a new Lax pair (Herman 1988). In this new Lax pair the time evolution equation is the linearized evolution equation, while associated spectral problem is related to the recursion operator, which generates the non-Lie point symmetries of the integrable equation (Fokas and Santini 1986, Herman 1990b). Using an expansion over the solutions of the new Lax pair, one can find the first-order correction u_1 as well as any changes in the shape and velocity of the solution of the unperturbed equation. The spectral part of this pair can be used to establish completeness and to provide the needed orthogonality relations for carrying out the method. Such a study has been started for the KP equation (Herman 1988) and results will be reported at a later date.

6. Summary

We have investigated the evolution of a modulated, planar KP soliton. This was done using the solution of a Cauchy problem for the linearized KP equation. Namely, we expanded the modulated initial condition

$$u(x,0) = 2\nu^2 \operatorname{sech}^2(\nu x + e^{i\omega y}) \tag{88}$$

and the solution to the KP equation

$$u = u_0 + \epsilon u_1 + O(\epsilon^2) = 2\nu^2 \operatorname{sech}^2 \nu(x - 4\nu^2 t) + \epsilon u_1. \tag{11}$$

Inserting this into the KP equation, we have transformed the problem into solving a linearized KP equation for $u_1(x,t)$, subject to the initial condition

$$u_1(x,0) = A_0 e^{i\omega y}(\operatorname{sech}^2\phi)_\phi \qquad A_0 = 2\nu^2 \qquad \phi = \nu x. \tag{30}$$

Using the general solution to this problem (27), we found that

$$u_1(x,y,t) = -\frac{A_0\omega^2}{8\nu^2} e^{i\omega y}\partial_\phi \int_{-\infty}^{\infty} \frac{dz}{z} e^{if(z)} \frac{\nu^2\tanh^2\phi - 2iz\nu\tanh\phi - kn}{(k^2+\nu^2)(n^2+\nu^2)\sinh(\pi z/\nu)}. \tag{61}$$

We studied the asymptotic behaviour of this solution both at the peak ($x = 0$), and for general values of x. Using these results, we have found that the initial modulations in the planar soliton decay, leaving behind a stable soliton. As the modulations decay, they tend to propagate away from the centre of the soliton with smaller secondary wavefonts following. Similar results have been seen in numerical experiments (Chang 1986, Chang *et al* 1986).

Finally, we have sketched a method for dealing with the stability of other solutions of the KP equation, as well as other integrable equations. This method relies on being able to obtain a new Lax pair for the equation under study, whose spectral and time evolution operators are the recursion operator and linearized time evolution operator,repectively. Applications of this general method to stablity and perturbation analyses are currently under way. The details will be reported at a later time.

Acknowledgments

This work was done at Clarkson University in Potsdam, NY, while the author was completing his dissertation. I wish to thank Dr D J Kaup for his reading of the manuscript and financial support. This work was completed under the NSF grant DMS-8501325 and grant AFOSR-86-0277.

Appendix 1. Failure of singular perturbation theory

In this section we review the original perturbation study of Kadomtsev and Petviashvili (1970). The authors turn their attention to the equation

$$u_t + 6uu_x + u_{xxx} = \varphi_y \tag{A1.1}$$

where φ_y is a small transverse correction to the KdV equation. The dependence on u of this particular term was determined as follows. Assume that u is a two-dimensional wave with a small amplitude, $u = \exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$, and a small wavelength in the x-direction. For this wave, moving at a velocity c in the x-direction, the oscillation frequency is given by

$$\omega = \mp(kc - k_x c) = \mp c\left(\sqrt{k_x^2 + k_y^2} - k_x\right) \simeq \mp\frac{c}{2}\frac{k_y^2}{k_x} \tag{A1.2}$$

where the upper (lower) sign refers to negative (positive) dispersion.

For this small-amplitude wave, the second and third terms in (A1.1) can be ignored. Thus, we find

$$\varphi_y \simeq -\mathrm{i}\omega u \simeq \mp\frac{c}{2}\mathrm{i}u\frac{k_y^2}{k_x}$$

implying

$$\varphi_{yx} \simeq \pm\frac{c}{2}k_y^2 u \simeq \mp\frac{c}{2}u_{yy}.$$

From this the relation between u and φ is

$$\varphi_x \simeq \mp\frac{c}{2}u_y. \tag{A1.3}$$

In view of this we can rewrite equation (A1.1) as

$$(u_t + 6uu_x + u_{xxx})_x + \alpha u_{yy} = 0 \qquad \alpha \equiv \pm\frac{c}{2} \tag{A1.4}$$

which is a familiar form of the so-called KP equation. (Note, in this discussion of the KP paper, we have adapted their equations to our notation and conventions, so that future comparisons can be easily made.)

We now sketch the perturbation analysis that Kadomtsev and Petviashvili used. Assuming that the transverse phase oscillation is very small, they solve equation (A1.1) by introducing the new variable $\phi = \eta(t)(x - x_0(t, y))$ to replace x in (A1.1), and by assuming that η, and x_0 are slowly varying parameters. Rewriting the derivatives as

$$\partial_x = \eta\partial_\phi \qquad \partial_t = \left(\frac{\eta_t}{\eta}\phi - \eta x_{0t}\right)\partial_\phi + \partial_t$$

equation (A1.1) becomes

$$(3\eta u^2 + \eta^3 u_{\phi\phi})_\phi = \varphi_y - u_t. \tag{A1.5}$$

Adding $-4\eta^3 u_\phi$ to both sides, and expanding u as

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots \qquad u_0 = 2\eta^2 \mathrm{sech}^2\phi \equiv 2\eta^2 v$$

they obtain in the first two orders

$$\eta^3\left(-4u_0 + \frac{3u_0^2}{\eta^2} + u_{0\phi\phi}\right)_\phi = 0 \tag{A1.6}$$

$$\eta^3\left(-4u_1 + 12u_1 v + u_{1\phi\phi}\right)_\phi = 2\eta^2(x_{0t} - 4\eta^2)v_\phi - 4\eta\eta_t[v + \tfrac{1}{2}\phi v_\phi] + \varphi_y. \tag{A1.7}$$

The first is essentially a KdV equation with the one-soliton solution

$$u_0 = 2\eta^2 v.$$

In the second equation, (A1.7), they assume that the last two terms are of higher order. Then the first-order equation in ϵ is given by

$$\eta^3\left(-4u_1 + 12u_1 v + u_{1\phi\phi}\right)_\phi = 2\eta^2(x_{0t} - 4\eta^2)v_\phi \tag{A1.8}$$

whose solution is given by

$$u_1 = \tfrac{1}{2}(x_{0t} - 4\eta^2)(v + \tfrac{1}{2}\phi v_\phi). \tag{A1.9}$$

u_1 can be made to vanish by choosing the time dependence of the phase as

$$x_{0t} = 4\eta^2. \tag{A1.10}$$

Now the second-order equation is

$$\eta^3\left(-4u_2 + 12u_2 v + u_{2\phi\phi}\right)_\phi = -4\eta\eta_t[v + \tfrac{1}{2}\phi v_\phi] + 2\alpha\eta^2 x_{0yy} v. \tag{A1.11}$$

Multiplying both sides of this equation by $\mathrm{sech}^2\phi$ and integrating over ϕ yields the condition

$$\eta_t = \tfrac{2}{3}\alpha\eta x_{0yy}. \tag{A1.12}$$

The conditions (A1.10) and (A1.12) can then be combined to give

$$x_{0tt} = \tfrac{16}{3}\eta^2\alpha x_{0yy}. \tag{A1.13}$$

This equation indicates that for $\alpha < 0$ (positive dispersion) solitary waves are unstable in the presence of transverse perturbations. For $\alpha > 0$ (negative dispersion) there are oscillations in the phase. Kadomtsev and Petviashvili noted that, in going to third order, it is found that these oscillations are damped.

However, implicit in this discussion is that to this order it was assumed that

$$\eta_y = 0. \tag{A1.14}$$

Differentiating (A1.12) with respect to y and using (A1.14), we find

$$x_{0yyy} = 0. \tag{A1.15}$$

The solutions for the phase must be of the form

$$x_0 = ay^2 + by + c. \tag{A1.16}$$

In the case that η is a constant, the solutions to (A1.13) are oscillatory (for $\alpha > 0$). This could be true for sufficiently small y, as (A1.16) could represent a series expansion about $y = 0$. It certainly is not adequate for studying the modulations in this paper.

This problem is not only present in the Kadomtsev–Petviashvili formulation. In the analysis presented by Chang (1986) and Chang *et al* (1986) they had arrived at equations (A1.10), (A1.12) and (A1.13) before going on to third order. So the same problem is present in their study.

Appendix 2. Steepest descent directions

Now that we have the saddle points in (45), we need to determine the direction of steepest descent. We look near the saddle points at points $\hat{z}$, such that

$$\hat{z} - z_j = \delta e^{i\alpha} \tag{A2.1}$$

where $\delta > 0$ is small and α is constant. Inserting this in (43) and Taylor expanding about z_j, we find that

$$\Delta_j \equiv f(\hat{z}) - f(z_j) = 2i\delta^2 e^{2i\alpha}(3z_j - C/z_j^3) + O(\delta^2). \tag{A2.2}$$

If we can write this in the form

$$\Delta_j = a e^{i(\theta+2\alpha)} + O(\delta^2) \qquad a > 0 \tag{A2.3}$$

then in the direction of steepest descent Δ_j is real and negative (Bleistein and Handelsman 1986). Thus

$$\theta + 2\alpha = (2p+1)\pi \qquad p = 0, 1. \tag{A2.4}$$

Similarly, the path of steepest ascent is given for Δ_j real and positive (Bleistein and Handelsman 1986). Therefore, the direction of steepest ascent is obtained from

$$\theta + 2\alpha = 2p\pi \qquad p = 0, 1. \tag{A2.5}$$

Now, all we have to do is to rewrite

$$2i\delta^2(3z_j - C/z_j^3)$$

as $\alpha e^{i\theta}$.

In order to make the computations a little easier, we note that in computing the poles and the saddle points, we have two parameters to work with: ω and ν. If we rescale z as $z = \nu z'$, which could have been done back in (17) and (18), we find that we need only one parameter, which we define as $\gamma = \omega/\nu^2$. The poles now take the form

$$\begin{aligned} &z = \pm\frac{i}{2}(1 \mp \sqrt{1 - i\gamma}) \qquad k = \pm i \\ &z = \pm\frac{i}{2}(1 \mp \sqrt{1 + i\gamma}) \qquad n = \pm i \\ &z = \pm im \qquad \sinh(\pi z) = 0 \end{aligned} \tag{A2.6}$$

and the stationary points are given by

$$z_j = (-1)^{j+1}\frac{i}{\sqrt{6}}\sqrt{1 + (-1)^{m+1}(1 - \tfrac{9}{4}\gamma^2)^{1/2}}. \tag{A2.7}$$

Returning to the computation of the paths of steepest descent, we find that there are two cases. If $\gamma < \frac{2}{3}$, then the saddle points are purely imaginary, while for $\gamma > \frac{2}{3}$, they are complex. We investigate these two cases separately.

For $\gamma < \frac{2}{3}$, the z_j are purely imaginary, so we can write

$$z_j = (-1)^{j+1}e^{i\pi/2}|z_j|. \tag{A2.8}$$

Inserting this in (A2.2), we compute

$$\begin{aligned} \Delta_j &= 2\delta^2 e^{i(2\alpha + \frac{\pi}{2})}(-1)^{j+1}\left(-i\frac{C}{|z_j|^3} + 3i|z_j|\right) \qquad C \equiv \frac{3\gamma^2}{16} \\ &= \frac{2\delta^2}{|z_j|^3}(-1)^{j+1}e^{2i\alpha}[C - 3|z_j|^4]. \end{aligned} \tag{A2.9}$$

We need to know the sign of $C - 3|z_j|^4$. In terms of C, we have

$$\begin{aligned} |z_j|^4 &= \tfrac{1}{36}[1 + (-1)^{m+1}\sqrt{1 - 12C}]^2 \\ &= \tfrac{1}{36}[2 - 12C + 2(-1)^{m+1}\sqrt{1 - 12C}]. \end{aligned} \tag{A2.10}$$

Therefore,

$$\begin{aligned} C - 3|z_j|^4 &= \tfrac{1}{6}[12C - 1 - (-1)^{m+1}\sqrt{1 - 12C}] \\ &= \tfrac{1}{6}[(-1)^{m+1}\sqrt{1 - 12C} - (1 - 12C)] \\ &= \frac{(-1)^m}{6}(1 - \tfrac{9}{4}\gamma^2)^{1/2}[1 + (-1)^{m+1}(1 - \tfrac{9}{4}\gamma^2)^{1/2}]. \end{aligned} \tag{A2.11}$$

For $\gamma < \frac{2}{3}$ the term in the brackets is always positive. So, we have

$$\operatorname{sgn}(C - 3|z_j|^4) = (-1)^m. \tag{A2.12}$$

This gives for (A2.2)

$$\Delta_j = \frac{2\delta^2}{|z_j|^3}|C - 3|z_j|^4|\mathrm{e}^{2\mathrm{i}\alpha}(-1)^{m+j+1}. \tag{A2.13}$$

From (A2.3) we can read off θ as

$$\theta = \begin{cases} \pi & m+j \text{ even} \\ 0 & m+j \text{ odd.} \end{cases} \tag{A2.14}$$

Using θ, we find that the extremal paths are given by

$$\alpha = (2p+1)\frac{\pi}{2} - \frac{\theta}{2} = \begin{cases} p\pi & m+j \text{ even} \\ p\pi + \frac{\pi}{2} & m+j \text{ odd} \end{cases} \quad (p = 0, 1). \tag{A2.15}$$

We now turn to the second case, $\gamma > \frac{2}{3}$. In this case we again want to evaluate (A2.2). Writing $z_j = R_j \mathrm{e}^{\mathrm{i}\phi_j}$, we find after some algebra that

$$R_j^4 = \frac{C}{3}.$$

Thus, (A2.2) yields

$$\Delta = \frac{2\delta^2}{R_j^3}\mathrm{e}^{\mathrm{i}(2\alpha + \frac{\pi}{2})}C(\mathrm{e}^{\mathrm{i}\phi_j} - \mathrm{e}^{3i\phi_j}). \tag{A2.16}$$

Using the identities

$$\cos 3\phi_j = 4\cos^3\phi_j - 3\cos\phi_j \qquad \sin 3\phi_j = 3\sin\phi_j - 4\sin^3\phi_j \tag{A2.17}$$

we have after some manipulation

$$\Delta_j = \frac{8\delta^2}{R_j^3}\sin\phi_j\cos\phi_j\mathrm{e}^{\mathrm{i}(2\alpha+\pi-\phi_j)}. \tag{A2.18}$$

We still need the sign of $\sin\phi_j\cos\phi_j$, but this just depends on which quadrant z_j is in. Defining P_j as

$$P_j = \begin{cases} 0 & z_j \text{ in quadrants I, III} \\ 1 & z_j \text{ in quadrants II, IV} \end{cases} \tag{A2.19}$$

we can write

$$\Delta_j = \frac{8\delta^2}{R_j^3}|\sin\phi_j\cos\phi_j|(-1)^{P_j}\mathrm{e}^{\mathrm{i}(2\alpha+\pi-\phi_j)}. \tag{A2.20}$$

We have from (A2.3)

$$\theta = \begin{cases} -\phi_j & P_j = 1 \\ -\phi_j + \pi & P_j = 0. \end{cases} \tag{A2.21}$$

The paths of steepest descent are then found from (A2.4) as

$$\alpha = \begin{cases} \frac{1}{2}\phi_j + p\pi & P_j = 0 \\ \frac{1}{2}\phi_j + (2p+1)\frac{\pi}{2} & P_j = 1. \end{cases} \tag{A2.22}$$

References

Ablowitz M J, Bar Yaacov D and Fokas A S 1983 On the inverse scattering transform for the Kadomtsev Petviashvili equation *Stud. Appl. Math.* **69** 135–43

Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

Bleistein N and Handelsman R H 1986 *Asymptotic Expansions of Integrals* (New York: Dover)

Boiti M, Leon J J-P, Martina L and Pempinelli F 1988 Scattering of localized solitons in the plane *Phys. Lett.* **132A** 432–9

Burtsev S P 1985 Damping of soliton oscillations in media with a negative dispersion law *Sov. Phys.–JETP* **61** 270–4

Chang H-Y 1986 On the excitation and propagation of ion acoustic solitons in nonideal plasmas *Dissertation* University of Iowa

Chang H-Y, Lien C, Hill J, Raychaudhuri S, Lonngren K E and Gabl E F 1986 On the two dimensional evolution of an ion acoustic soliton *Phys. Fluids* **29** 3769–74

Fokas A S and Ablowitz M J 1983 On the inverse scattering of the time dependent Schrödinger equation and the associated Kadomtsev–Petviashvili (I) equation *Stud. Appl. Math.* **69** 211–28

—— 1984 On the inverse scattering transform of multidimensional nonlinear equations related to first order systems in the plane *JMP* **25** 2494–505

Fokas A S and Santini P M 1986 The recursion operator of the Kadomtsev–Petviashvili equation and the squared eigenfunctions of the Schrödinger operator *Stud. Appl. Math.* **75** 79–85

—— 1989 Coherent structures in multidimensions *Phys. Rev. Lett.* **63** 1329–33

Gabl E F, Bulson J M and Lonngren K E 1984 Excitation of a two dimensional ion acoustic soliton *Phys. Fluids* **27** 269–72

Herman R L 1988 Soliton perturbations, revisited *Dissertation* Clarkson University

—— 1990a A direct approach to studying soliton perturbations *J. Phys. A: Math. Gen.* **23** 2327–62

—— 1990b Lax pairs, recursion operators and the perturbation of nonlinear evolution equations *Proc. AMS Summer Research Conf. on Inverse Scattering and Applications* to appear

Hietarinta J 1990 One-dromion solution for generic classes of equations *Phys. Lett.* **149A** 113–8

Kadomtsev B B and Petviashvili V I 1970 On the stability of solitary waves in weakly dispersive media *Sov. Phys.–Dokl.* **15** 539–41

Kako M and Rowlands G 1976 Two dimensional stability of ion acoustic solitons *Plasma Phys.* **18** 165–70

Lonngren K E 1983 Soliton experiments in plasmas *Plasma Phys.* **25** 943–82

Lonngren K E, Gabl E F, Bulson J M and Khazei M 1983 On the resonant interaction of spherical ion acoustic solitons *Physica* **9D** 372–8

Washimi H and Taniuti T 1966 Propagation of ion acoustic solitary waves of small amplitude *Phys. Rev. Lett.* **17** 996–8

Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)